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PATH-CLOSED SETS

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Given a digraph $G=(V, E)$, call a node set $T\subseteq V$ path-closed if $v, v'\in T$ and $w\in V$ is on a path from v to v' implies $w\in T$. If G is the comparability graph of a poset P, the path-closed sets of G are the convex sets of P. We characterize the convex hull of (the incidence vectors of) all path-closed sets of G and its antiblocking polyhedron in $\mathbf{R}^V$, using lattice polyhedra, and give a min-max theorem on partitioning a given subset of V into path-closed sets. We then derive good algorithms for the linear programs associated to the convex hull, solving the problem of finding a path-closed set of maximum weight sum, and prove another min-max result closely resembling Dilworth's theorem.

1. Introduction

A typical program in polyhedral combinatorics is: given a combinatorial optimization problem with linear objective, find a linear programming formulation of it [2]. This requires the description of the convex hull of the solution vectors as a solution set of a system of linear inequalities, a task which can sometimes be very difficult. However, if successful, it will not only enrich the family of integer polyhedra, but often yield combinatorial min-max theorems and further or include the development of good algorithms for the initial problem at hand.

Consider a directed graph $G=(V, E)$ with node set V and edge set E, and call a set $T\subseteq V$ *path-closed* if $v, v'\in T$ and $w\in V$ is on a directed path from v to v' implies $w\in T$. Let an objective function $c\colon V\to\mathbf{R}$ attribute a real weight c_v to each node $v\in V$. We shall carry out the above program for the following combinatorial optimization problem: find a path-closed set T of maximum weight $\Sigma\{c_v\colon v\in T\}$.

First some preliminary remarks: throughout the paper, the graphs, paths and circuits mentioned shall be directed, and, for simplicity and from now on, not explicitly qualified as such. Some standard notation shall also be used, in particular for any set A, $z\in\mathbf{R}^A$ and $B\subseteq A$, $z(B)$ denotes the sum $\Sigma\{z_a\colon a\in B\}$, $\bar{B}$ is the complement set of B, and a, $a\in E$, is sometimes written for the singleton $\{a\}$. In the graph $G=(V, E)$, $t(e)$ and $h(e)$ are the tail and head of edge e, (v, w) the edge with tail v and head w, and for any $S\subseteq V$, $\delta(S)=\{e\in E\colon t(e)\in S,\ h(e)\in\bar{S}\}$.

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Observe that the notion of a path-closed set of a graph G is close to that of a *convex set* of a poset, and in fact equivalent if G contains no circuits. If P is a poset with ordering $\leqq$, a set $T \subseteq P$ is convex if $a, d \in T$, $b \in P$ and $a \leqq b \leqq d$ implies $b \in T$. The *comparability graph* $\hat{G}$ of P is the graph with node set P and edge set $\hat{E}$ such that $(a, b) \in \hat{E}$ iff $b < a$. $\hat{G}$ contains no circuits and the path-closed sets of $\hat{G}$ are the convex sets of P. On the other hand, given any graph $G=(V, E)$ without circuits, its path-closed sets are the convex sets of the poset whose comparability graph is the *transitive closure* $\hat{G}=(V, \hat{E})$ of G ($(v, w) \in \hat{E}$ iff there is a path from v to w in G).

The different steps of the program and the plan of the paper are the following.

First, we find the convex hull of (the incidence vectors of) all path-closed sets of G via its *antiblocking polyhedron in* $\mathbf{R}^V$. More specifically, let $L=\{T \subseteq V: T$ path-closed$\}$, and for any $T \in L$, let $\bar{A}_T \in \mathbf{R}^V$ be the incidence vector of T and $\bar{A}$ the matrix with rows $\bar{A}_T$, $T \in L$. Further, for any matrix M, denote by $\mathrm{CONV}(M)$ the convex hull of its row vectors and by $\mathrm{CONE}(M)$ the cone generated by its row vectors. In Sections 2 and 3, we shall show, using lattice polyhedra, that

$$(1.1) \qquad \{x \in \mathbf{R}^V \colon \bar{A}x \leqq 1\} = \mathrm{CONV}(A) + \mathrm{CONE}(R) - \mathbf{R}^V_+$$

for some 0, ± 1-matrices A and R, which we characterize, and deduce from it that

$$(1.2) \qquad \{x \in \mathbf{R}^V_+ \colon Ax \leqq 1,\ Rx \leqq 0\} = \mathrm{CONV}(\bar{A}).$$

(As we shall see, there is some legitimacy in calling the polyhedron of (1.1) the "antiblocking polyhedron" of $\mathrm{CONV}(\bar{A})$). As a by-product, we derive a min-max result on partitioning a given subset of V into a minimum number of path-closed sets.

In Section 4, we give good algorithms for the linear programs associated with the left side of (1.2), solving in particular the problem of finding a path-closed set of maximum weight. The algorithms are based on the determination of a min flow and a max cut in some auxiliary graph. We then derive a further min-max theorem resembling Dilworth's theorem, and conclude with a remark on the polyhedron of (1.2).

2. Path-closed sets and lattice polyhedra

Let us briefly summarize some definitions and results on integer polyhedra and lattice polyhedra.

A polyhedron $P \subseteq \mathbf{R}^n$ is said to be an *integer polyhedron* if each non empty face of P contains an integer valued point. In particular, if P is integer, each vertex of P is integer valued.

Given rational $m \times n$-matrix A and m-vector b, the system of linear inequalities $Ax \leqq b$ is said to be *total dual integral* (tdi) if for any integer valued vector c such that the linear program (lp) "maximize cx s.t. $Ax \leqq b$" has an optimal solution, its dual lp "minimize by s.t. $y \geqq 0$, $yA=c$" has an *integer* optimal solution.

Theorem 2.1. [12, 3]. *A sufficient condition for $P=\{x \in \mathbf{R}^n\colon Ax \leqq b\}$ to be an integer polyhedron is that the system $Ax \leqq b$ is tdi and b is integer valued.* ∎

Total dual integrality has shown to be a powerful tool for proving that a polyhedron is integer, is due to Hoffman [12] and has been generalized to the above form by Edmonds and Giles [3]. Related work is also found in [6, 17].

Lattice polyhedra are a certain class of integer polyhedra. They are given as solutions sets of systems of linear inequalities which are tdi. Introduced in [15] by Hoffman and Schwartz, they have been further developed in [13, 14, 10, 11]. We use here the notation of [11].

Let $\hat{L}$ be a finite lattice with partial order $\leqq$, meet $\wedge$ and join $\vee$. The real valued function $h: \hat{L} \to \mathbf{R}$ is *submodular* if for all $a, b \in \hat{L}$, $h(a \vee b) + h(a \wedge b) \leqq h(a) + h(b)$, *supermodular* if the inequality is reversed and *modular* if equality holds. The 0, ± 1-valued function $f: \hat{L} \to \{0, \pm 1\}$ is *consecutive* if $a, b \in \hat{L}$ with $a \leqq b$ implies $|f(b) - f(a)| \leqq 1$, and $a, b, c \in \hat{L}$ with $a < b < c$ implies $|f(a) - f(b) + f(c)| \leqq 1$. One of the results of [11] is

Theorem 2.2. *Given a submodular function $r: \hat{L} \to \mathbf{Z}$, n consecutive and modular functions $f_j: \hat{L} \to \{0, \pm 1\}$, $j = 1, \ldots, n$, and vectors $d, e \in \{\mathbf{Z} \cup \pm\infty\}^n$, the polyhedron*

$$\{x \in \mathbf{R}^n: e \leqq x \leqq d, \ \sum f_j(a) x_j \leqq r(a): a \in \hat{L}\}$$

is integer and the given system describing it is tdi. ▮

Returning now to our family L of all path-closed sets of G and its incidence matrix $\bar{A}$ introduced in Section 1, we prove:

Theorem 2.3. *The polyhedron $\mathscr{A}(\bar{A}) \equiv \{x \in \mathbf{R}^V: \bar{A}x \leqq 1\}$ is integer and the describing system is tdi.*

Proof. We show that $\mathscr{A}(\bar{A})$ is essentially a lattice polyhedron of the type given in Theorem 2.2. For brevity, let us introduce the binary relation $\leqq$ defined on $V \times V$ by $w \leqq v$ if there exists a path in G from v to w, $v, w \in V$. Let $C \subseteqq V$ be said to be *closed above* if for any $v \in C$ and $w \in V$, $v \leqq w$ implies $w \in C$. Analogously, $D \subseteqq V$ is said to be *closed below* if for any $v \in D$ and $w \in V$, $w \leqq v$ implies $w \in D$. Let $\bar{L} = \{C \subseteqq V$: C closed above$\}$ and $\underline{L} = \{D \subseteqq V$: D closed below$\}$. Clearly, for any $C, C' \in \bar{L}$, $C \cap C'$ and $C \cup C' \in \bar{L}$ and for any $D, D' \in \underline{L}$, $D \cap D'$ and $D \cup D' \in \underline{L}$. Let $F \subseteqq \bar{L} \times L$ be defined by

$$F = \{(C, D): C \in \bar{L}, \ D \in \underline{L} \text{ and } C \cup D = V\}$$

The following ordering and meet and join operations define a lattice on F:

$$(C, D) \leqq (C', D') \quad \text{if} \quad C \subseteqq C' \quad \text{and} \quad D \supseteqq D'$$

$$(C, D) \wedge (C', D') = (C \cap C', D \cup D')$$

$$(C, D) \vee (C', D') = (C \cup C', D \cap D').$$

For all $v \in V$, define $f_v: F \to \{0, 1\}$ by

$$f_v(C, D) = \begin{cases} 1 & \text{if } v \in C \cap D \\ 0 & \text{otherwise} \end{cases} \quad \text{for any} \quad (C, D) \in F.$$

It is easy to verify that the functions f_v, $v \in V$, are consecutive and modular on F. Therefore, by Theorem 2.2, the polyhedron

$$\{x \in \mathbf{R}^V: x(C \cap D) \leqq 1, \ (C, D) \in F\} \tag{2.1}$$

has all its vertices integer valued and its system is tdi.

We show now that the polyhedron (2.1) is $\mathscr{A}(\bar{A})$. Clearly, for any $(C, D)\in F$, $T=C\cap D$ is path-closed. Conversely, any $T\in L$ can be represented, although not uniquely, by $C\cap D$ for some $(C, D)\in F$. To see this, let $C=\{v\in V: w\leqq v$ for some $w\in T\}$ and $D'=\{v\in V: v\leqq w$ for some $w\in T\}$. Clearly, $T\subseteq C\cap D'$, and $C\in\bar{L}$, $D'\in\underline{L}$. Also, if $v\in C\cap D'$, then $w\leqq v\leqq u$ for some $w, u\in T$, therefore $v\in T$ and $T=C\cap D'$. If $C\cup D'=V$, $(C, D')\in F$ and we are done. If $C\cup D'\neq V$, let $D''=\{v\in V: v\leqq w$ for some $w\in V-(C\cup D')\}$. Clearly, $D''\supseteq V-(C\cup D')$ and $D''\in\underline{L}$. Therefore $D'\cup D''\in\underline{L}$ and $C\cup(D'\cup D'')=V$. Furthermore, $C\cap D''=\emptyset$, as from $v\in C$ follows $w\leqq v$ for some $w\in T$, from $v\in D''$, $v\leqq u$ for some $u\in V-(C\cap D')$, hence $w\leqq u$, implying $u\in C$, a contradiction. We have in turn $T=C\cap D$, where $C\in\bar{L}$ and $D=D'\cup D''\in\underline{L}$ and $C\cup D=V$, i.e. $(C, D)\in F$. Therefore the constraints in (2.1) are $\bar{A}x\leqq 1$, up to possible duplication of some of them; hence (2.1) is $\mathscr{A}(\bar{A})$ and $\bar{A}x\leqq 1$ is tdi. ▮

3. Alternating vectors of paths and circuits

Let P be a path of G with node set $V(P)$. The 0, ± 1-vector $A_0\in\mathbf{R}^V$ will be said to be an *alternating vector of P in G* if its support is in $V(P)$, and, traversing P and ignoring the zero components, the $+1$ and -1 alternate in a sequence $+1$, -1, $+1$, ..., -1, $+1$. Analogously, if Q is a circuit of G with node set $V(Q)$, the 0, ± 1-vector $R_0\in\mathbf{R}^V$ will be said to be an *alternating vector of Q in G* if its support is in $V(Q)$ and, traversing Q and ignoring the zero components, the $+1$ and -1 alternate in an sequence $+1$, -1, ..., $+1$, -1.

Let A_i, $i\in I$, be all alternating vectors of paths of G, R_j, $j\in J$, all alternating vectors of circuits of G, and A and R the matrices with rows A_i, $i\in I$, and R_j, $j\in J$. We prove now (1.1) and (1.2) mentioned in the introduction.

Theorem 3.1. a) $\mathscr{A}(\bar{A})\equiv\{x\in\mathbf{R}^V: \bar{A}x\leqq 1\}=\mathrm{CONV}(A)+\mathrm{CONE}(R)-\mathbf{R}^V_+$,
b) $\mathscr{A}(A, R)\equiv\{x\in\mathbf{R}^V_+: Ax\leqq 1,\ Rx\leqq 0\}=\mathrm{CONV}(\bar{A})$.

Proof of a). We proceed in two steps, assuming in the first that G contains no circuits.

i) As G has no circuits, the binary relation $\leqq$ defined previously on V is a partial order, making of V a poset. It is with reference to this poset that we use below the terms incomparable, chain, antichain etc. Now any alternating vector of a path, A_i, is in $\mathscr{A}(\bar{A})$ and is easily shown to be a vertex of $\mathscr{A}(\bar{A})$. So let us prove that any vertex $\bar{x}$ of $\mathscr{A}(\bar{A})$ is such an A_i.

Clearly, $\bar{x}\leqq 1$. Also, for any $T\in L$ and $v\in T$, letting $C=\{w\in T: v<w\}$, $D=\{w\in T: w<v\}$ and $N=\{w\in T: v$ and w incomparable$\}$, we have $T=C\cup(D\cup N)\cup\{v\}$ and C, D and $D\cup N\in L$. Therefore, for any $x\in\mathscr{A}(\bar{A})$, $\bar{A}_Tx=x(T)=x(C)+x(D\cup N)+x_v\leqq 2+x_v$. Hence, for any maximal $x\in\mathscr{A}(\bar{A})$, in particular $\bar{x}$, $x\geqq -1$. Therefore, by Theorem 2.3, $\bar{x}$ is 0, ± 1-valued. The following arguments show that $\bar{x}$ is an alternating vector A_i. $\{v\in V: \bar{x}_v=1\}$ is a chain, say $\{v_1, ..., v_s\}$ with $v_1<...<v_s$, as any antichain is a member of L. Also, there exists $\{w_1, ..., w_{s-1}\}\subseteq\{v\in V: \bar{x}_v=-1\}$ such that $v_1<w_1<v_2<...<w_{s-1}<v_s$, as any interval $\{v: v_r\leqq v\leqq v_{r+1}\}$, $r=1, ..., s-1$, is a member of L. $\{w_1, ..., w_{s-1}\}=\{v\in V: \bar{x}_v=-1\}$ follows from the maximality of $\bar{x}$ in $\mathscr{A}(\bar{A})$: let $v\in V$ such that $\bar{x}_v=-1$ and $T\in L$

such that $v \in T$ and $\bar{x}(T)=1$. Clearly, the set $\{v_1, ..., v_s\} \cap T$ is non-empty and is a chain. Let v_α and v_β be its minimal and maximal elements. From $T \in L$ follows $C=\{v_\alpha, w_\alpha, ..., w_{\beta-1}, v_\beta\} \subseteq T$. Hence $\bar{x}(T)=\bar{x}(C)+\bar{x}(T-C)=1$, and from $\bar{x}(C)=1$ and $\bar{x}(T-C) \leqq 0$ follows $\bar{x}_v=0$ for all $v \in T-C$; therefore $v \in \{w_\alpha, ..., w_{\beta-1}\}$.

As the recession cone of $\mathscr{A}(\bar{A})$ is $-\mathbf{R}^V_+$, we have proven $\mathscr{A}(\bar{A})=\text{CONV}(A)-\mathbf{R}^V_+$.

ii) Assuming now that G contains circuits, we proceed by induction on the number of circuits.

Let Q be a minimal circuit of G with node set $V(Q)$ and contract Q, obtaining $G'=(V', E')$ with pseudo-node v_q and family L' of path-closed sets. Denote for any $T \in L'$ by $\bar{A}'_T \in \mathbf{R}^{V'}$ the incidence vector of T. As for any $T \in L$, either $V(Q) \cap T=\emptyset$ or $V(Q) \subseteq T$, the system $\bar{A}_T x \leqq 1$, $T \in L$, is readily obtained from the system $\bar{A}'_T x \leqq 1$, $T \in L'$, by duplicating the column v_q $|V(Q)|-1$ times. This operation corresponds to expanding v_q in G' to recover G. We now state an easy lemma, the proof of which is left to the reader:

Lemma 3.2. *Let $\mathscr{P}'=\{x' \in \mathbf{R}^n: M'x' \leqq b\}$ be a non empty polyhedron. Let A', Γ' be matrices such that $\mathscr{P}'=\text{CONV}(A')+\text{CONE}(\Gamma')$, and duplicate in $M'x' \leqq b$ the n-th column of M' and x'_n m times, obtaining $Mx \leqq b$. Then $\mathscr{P}=\{x \in \mathbf{R}^{n+m}: Mx \leqq b\}$ is such that $\mathscr{P}=\text{CONV}(A)+\text{CONE}(\Gamma)+\text{CONE}(\Delta^k: k=1, ..., 2m)$, where the rows of the matrices A and Γ agree with the rows of A' and Γ' in their n first components and their m last components are zero, and where for $k=1, ..., m$, the only non zero components of the vector Δ^k are $\Delta^k_n=1$ and $\Delta^k_{n+k}=-1$, and for $k=m+1, ..., 2m$, $\Delta^{m+k}=-\Delta^k$.* ▮

Assuming now for G' that $\mathscr{A}(\bar{A}')=\text{CONV}(A')+\text{CONE}(R')-\mathbf{R}^{V'}_+$, where the rows of A' and R' are the alternating vectors of the paths and circuits of G', expand v_q and apply the lemma for $M'=\bar{A}'$, $A'=A'$ and $\Gamma'=\begin{bmatrix} R' \\ -I' \end{bmatrix}$, where I' is the unit matrix of size $|V'|$. Clearly, the rows of A and Γ are the rows of A and $\begin{bmatrix} R \\ -I \end{bmatrix}$ and the Δ^k are alternating vectors of Q. Therefore $\mathscr{A}(\bar{A})$ is contained and hence equal to $\text{CONV}(A)+\text{CONE}(R)-\mathbf{R}^V_+$.

Proof of b). Clearly, $\text{CONV}(\bar{A}) \subseteq \{x \in \mathbf{R}^V_+: Ax \leqq 1, \quad Rx \leqq 0\} \equiv \mathscr{A}(A, R)$. Also, if we define $\mathscr{A}(\mathscr{A}(\bar{A})) \equiv \{x \in \mathbf{R}^V_+: xz \leqq 1 \text{ for all } z \in \mathscr{A}(\bar{A})\}$, then $\mathscr{A}(\mathscr{A}(\bar{A}))=\mathscr{A}(A, R)$ by a), and it is easy to show that $x \notin \text{CONV}(\bar{A})$ implies $x \notin \mathscr{A}(\mathscr{A}(\bar{A}))$; therefore $\text{CONV}(\bar{A}) \supseteq \mathscr{A}(A, R)$. Observe that $\bar{A}$ in assertion b) must contain the zero row, the incidence vector of the trivial path-closed set $\emptyset$. ▮

Remark 3.3. The relation between $\mathscr{A}(\bar{A})$ and $\mathscr{A}(A, R)$ is reminiscent of the antiblocking relation introduced by Fulkerson [4, 5]. In fact, it is possible to develop a general antiblocking relation (using two cones and their reverse polar cones) which subsumes both types of relations mentioned (as well as the one of Johnson [16]), and for which the expected properties (min-max equality, max-max inequality etc.) hold [7]. $\mathscr{A}(\bar{A})$ and $\mathscr{A}(A, R)$ are in this sense an antiblocking pair with respect to $\mathbf{R}^n, \mathbf{R}^n_+$.

A corollary to theorems 2.3 and 3.1 a) is the following min-max result. Given any node subset S of a graph G, call a path in G an *S-path* of G if its starting and terminal nodes are both in S.

Corollary 3.4. *Given a graph $G=(V, E)$ and $S \subseteq V$ such that for any circuit Q of G with node set $V(Q)$, either $V(Q) \cap S = \emptyset$ or $V(Q) \subseteq S$, the minimum number of path-closed sets of G which partition S is equal to the maximum number of times that an S-path leaves S plus one.*

Proof. Consider the linear program (lp)

$$\text{(3.1)} \qquad \text{maximize } dx, \quad x \in \mathbf{R}^V, \quad \text{s.t.} \quad \bar{A}x \leqq 1,$$

and its dual lp. Let d be the incidence vector of S, and interpret the duality theorem fulfilled by a solution to (3.1) being an alternating vector of a path (of minimal support) and a dual solution which is integer, hence 0,1-valued. ∎

Corollary 3.4 can also be derived from the polar version of Dilworth's theorem (or, more accurately from its proof), as has been pointed out by a referee. Clearly, min ≧ max in the corollary. Assume S to contain no circuits (contracting circuits always allows to reduce the general case to this situation), and define a partial order on S by $v \geqq w$, $v, w \in S$, if there is a path from v to w using at least one node not in S, or if $v = w$. In this poset, denoted $\hat{S}$, the length of a longest chain is equal to the minimum number of antichains partitioning $\hat{S}$. This minimum partition can be obtained in the usual way, by repeatedly identifying the set of minimal elements and deleting it from $\hat{S}$. It is easy to show that such a partition *is* a partition of S into path-closed sets of G, and, clearly, any chain of length, say, $\alpha+1$ is an S-path in G leaving S α times, proving the corollary. Observe that not any antichain of $\hat{S}$ is a path-closed set of G, neither is any minimum partition of $\hat{S}$ into antichains a partition of S into path-closed sets in G.

4. Optimum path-closed sets

Given a graph $G=(V, E)$ and a weight vector $c \in \mathbf{R}^V$, the problem of finding in G a path-closed set T of maximum weight sum $c(T) = \Sigma\{c_v : v \in T\}$ is equivalent by theorem 3.1b) to the lp "maximize cx, s.t. $Ax \leqq 1$, $Rx \leqq 0$, $x \geqq 0$". In order to solve the optimum path-closed set problem, we solve therefore this lp with (the incidence vector x^T of) some path-closed set T. Clearly, any x^T, $T \in L$, is feasible therein. To prove optimality, we will construct a solution to the dual lp with the same objective value, solving this dual lp as well.

In order to simplify the exposition, we assume G to contain no circuits and shall deal with the general case in Remark 4.5. We consider therefore the following lp's:

$$\text{(4.1)} \qquad \text{maximize } cx, \quad x \in \mathbf{R}^V, \quad \text{s.t.} \quad Ax \leqq 1, \quad x \geqq 0, \quad \text{and}$$

$$\text{(4.2)} \qquad \text{minimize } 1y, \quad y \in \mathbf{R}^I, \quad \text{s.t.} \quad yA \geqq c, \quad y \geqq 0,$$

The algorithms to be presented for (4.1) and (4.2) are based on the determination of a min flow and a max cut in an auxiliary graph. Both their formulation and proof of their validity are simpler if we consider a *node constraint* network, where capacities are given on nodes instead of edges. So let us make the following digression on the minimum flow problem in such a network.

Let $\hat{G}=(\hat{V},\hat{E})$ be a graph, $s,t\in\hat{V}$ two distinguished nodes and $l,u\in\in(\mathbf{R}_+\cup\{+\infty\})^{\hat{V}}$ two vectors attributing a non negative lower and upper bound to each node. Define an $s-t$-flow of value σ to be a vector $x\in\mathbf{R}_+^{\hat{E}}$ such that $l_v\leqq\leqq x(\delta(v))\leqq u_v$, $v\in\hat{V}$, and $x(\delta(\bar{v}))-x(\delta(v))=0$ for $v\in\hat{V}-\{s,t\}$, and $=-\sigma(+\sigma)$ for $v=s$ $(v=t)$. For any $S\subseteqq\hat{V}$, define

$$A_S^+ = \{v\in S\colon \nexists w\in S \text{ with } (v,w)\in\hat{E}\}$$

$$A_S^- = \{v\in\bar{S}\colon \exists w\in S \text{ with } (v,w)\in\hat{E}\}.$$

(Sets S, A_S^+, A_S^-, or more accurately, vectors $f^S\in\mathbf{R}^{\hat{V}}$ with components $f_v^S=1$ if $v\in A_S^+$, $=-1$ if $v\in A_S^-$ and $=0$ otherwise, have been introduced in an application of lattice polyhedra and appropriately associated with node cut sets [11]). It is not difficult to show the following version of the Min Flow-Max Cut theorem, as well as the corollary:

Theorem 4.1. [8] *If $\hat{G}$ admits an $s-t$-flow, the minimum value β of such a flow satisfies* $\beta=\max\{l(A_S^+)-u(A_S^-)\colon S\subseteqq V, s\in S\not\ni t\}$. ∎

Let a maximizing S above be called a *max cut.*

Corollary 4.2. *If x is a min flow and S a max cut, then*

(4.3) $$x(\delta(v)) = l_v \quad \textit{for any} \quad v\in A_S^+$$

$$x(\delta(v)) = u_v \quad \textit{for any} \quad v\in A_S^-$$

(4.4) $$x_{(v,w)} = 0 \quad \textit{for any} \quad v\in(S-A_S^+)\cup A_S^- \quad \text{and} \quad w\in\bar{S}. \quad ∎$$

We close our digression by observing that good algorithms are available for determining a min flow and a max cut in a usual (edge constraint) network. Adaptations of them or their application on a suitably derived network will provide a min flow and max cut of theorem 4.1.

Without loss of generality, assume that $c_v\neq 0$ for all $v\in V$ and $c_v>0$ for at least one $v\in V$. We can now state the algorithms for (4.1) and (4.2):

Primal Algorithm: Let $V^+=\{v\in V\colon c_v>0\}$, $V^-=\{v\in V\colon c_v<0\}$ and construct the following node constraint network: $\hat{G}=(\hat{V},\hat{E})$, where

$$\hat{V} = V\cup\{s,t\},$$

$$\hat{E} = \{(s,v),(v,t)\colon v\in V^+\}\cup\{(v,w)\colon (v\in V^+, w\in V^- \text{ or } v\in V^-, w\in V^+), \text{ and } v>w\},$$

and with lower and upper bounds

$$l_v = \begin{cases} c_v & \text{for } v\in V^+ \\ 0 & \text{for } v\in\{s,t\}\cup V^- \end{cases} \qquad u_v = \begin{cases} -c_v & \text{for } v\in V^- \\ \infty & \text{for } v\in\{s,t\}\cup V^+. \end{cases}$$

Determine an $s-t$-flow x of minimum value β and a max cut S. The set $T==(A_S^+\cap V^+)\cup A_S^-$ is a path-closed set of G of maximum weight sum β.

Dual Algorithm:

Step 0. Start with an $s-t$-flow x of minimum value β in $\hat{G}$. Set $y_i=0$ for all $i\in I$.

Step 1. If $x=0$, stop: y is optimal in (4.2). Else go to Step 2.

Step 2. Determine an $s-t$-path $\hat{P}$ with node set $V(\hat{P})$ and edge set $E(\hat{P})$ such that $x_e>0$ for all $e\in E(\hat{P})$, and $\varepsilon=\min\{x_e:\ e\in E(\hat{P})\}$. Reduce x_e by ε for all $e\in E(\hat{P})$. The vector $b\in\mathbf{R}^V$ with components

$$b_v=\begin{cases} 1 & \text{for } v\in V^+\cap V(\hat{P}),\\ -1 & \text{for } v\in V^-\cap V(\hat{P}),\\ 0 & \text{otherwise}\end{cases}$$

is an alternating vector, say A_i, of a path of G. Set $y_i=\varepsilon$ and go to Step 1.

Proof of the validity of the algorithms. Clearly, there exists a minimum $s-t$-flow x, since there is a flow $\hat{x}$ given by $\hat{x}_{(s,v)}=\hat{x}_{(v,t)}=l_v$ for all $v\in V^+$, $\hat{x}_e=0$ for all other $e\in\hat{E}$.

Let us first look at the dual algorithm and consider an application of Step 2. When an $s-t$-path $\hat{P}$ of $\hat{G}$ is found, some corresponding path P of G traversing all intermediate nodes of $\hat{P}$, in the same order as $\hat{P}$, is easily identified, since $(v, w)\in\hat{E}$ implies that there is a path from v to w in G. Moreover, since the edges of $\hat{G}$ connect only nodes of V^- with nodes of V^+, and s and t with nodes of V^+, the vector b is an alternating vector of P. It should then be clear from the decomposition of the minimum flow of value β and the choice of the upper and lower bounds u_v and l_v that $y\in\mathbf{R}^I$ generated by the dual algorithm is feasible in (4.2) and has the value $1y=\beta$.

Further, S being a max cut, $\beta=l(\Lambda_S^+)-u(\Lambda_S^-)$. By the choice of the u's and l's, $\Lambda_S^-\subseteq V^-$, $-u(\Lambda_S^-)=c(\Lambda_S^-)$, $l(\Lambda_S^+)=l(\Lambda_S^+\cap V^+)=c(\Lambda_S^+\cap V^+)$, so that $\beta=c(T)$. It remains to show the primal feasibility in (4.1) of the incidence vector of T, i.e. that T is path-closed.

First, $s\notin\Lambda_S^+$ and $t\notin\Lambda_S^-$ as $l(\Lambda_S^+)>0$ and $u_t=\infty$. Also, S being a max cut, we may assume for the rest of the proof that $\Lambda_S^+\cap V^-=\emptyset$ (otherwise, replace S by $S-(\Lambda_S^+\cap V^-)$ which is again a max cut by the choice of the l's). So we prove that $T=\Lambda_S^+\cup\Lambda_S^-$ is path-closed, where $\Lambda_S^+\subseteq V^+$, $\Lambda_S^-\subseteq V^-$ and S is a max cut. Let $C=\{v\in V:\ v\geqq w$ for some $w\in\Lambda_S^+\}$ and $D=\{v\in V:\ v\leqq w$ for some $w\in\Lambda_S^+\}$. Clearly, $C\cap D$ is path-closed. We show that $T=C\cap D$. Obviously, $\Lambda_S^+\subseteq C\cap D$. Also, for any $v\in\Lambda_S^-$, by $u_v>0$ and (4.3), there is an $s-t$-path $\hat{P}$ with edge set $E(\hat{P})$ going through v and such that $x_e>0$ for all $e\in E(\hat{P})$. The last node v_1 in S on the subpath of $\hat{P}$ from s to v must be in Λ_S^+ by (4.4), and similarly for the last node v_2 in $\bar{S}$ on the subpath of $\hat{P}$ from v to t. Then $v_1>v>v_2$ and $v_1, v_2\in C\cap D$ imply $v\in C\cap D$, therefore $T\subseteq C\cup D$. Next, take some $v\in C\cap D$. There are $v_1, v_2\in\Lambda_S^+$ with $v_1\geqq v\geqq v_2$. Suppose $v\in V^-$. Then $v\notin\Lambda_S^+$, and since $(v_1, v)\in\hat{E}$, $v\notin S$, so that $(v, v_2)\in\hat{E}$ implies $v\in\Lambda_S^-$. On the other hand, if $v\in V^+$, either $v\in S-\Lambda_S^+$, or $v\in\bar{S}-\Lambda_S^-$ or $v\in\Lambda_S^+$. In the first case, by $l_v>0$, there is $e\in\hat{E}$ with $t(e)=v$ and $x_e>0$. Then $h(e)\in S-\Lambda_S^+$ by $h(e)\in V^-$ and (4.4), and $v_1>v>h(e)$ contradicts $v_1\in\Lambda_S^+$. In the second case, there is $e\in\hat{E}$ with $h(e)=v$ and $x_e>0$, and $t(e)\in V^-$ and (4.4) imply $t(e)\in\bar{S}-\Lambda_S^-$, but then $t(e)>v>v_2$ contradicts $t(e)\notin\Lambda_S^-$. Therefore v must be in Λ_S^+ and $C\cap D\subseteq T$, hence $C\cap D=T$. ∎

For any $a \in \mathbf{R}^V$, let a^+ and a^- be the positive and negative support sets of a, i.e. $a^+ \equiv \{v \in V\colon a_v > 0\}$ and $a^- \equiv \{v \in V\colon a_v < 0\}$, and, for any $a, b \in \mathbf{R}^V$, say that *a conforms to b* if $a^+ \subseteqq b^+$ and $a^- \subseteqq b^-$. A by-product of the dual algorithm is the following

Theorem 4.3. *There is an optimal solution* $y \in \mathbf{R}^I$ *to (4.2)*

(i) *which is integer if c is integer,*

(ii) *such that the rows* A_i, $i \in I$ *and* $y_i > 0$, *all conform to c.* ∎

An immediate corollary is then a min-max result which is best formulated in terms of posets. So, let P be a poset with ordering $\leqq$. Given any $S \subseteqq P$, say that a family of subsets of P *covers* S if any $a \in S$ is in some member of the family, and *is packed in* S if no $a \in S$ is in more than one member of the family. Also, given a partition P^+, P^- of P with $P^+ \cup P^- = P$, $P^+ \cap P^- = \emptyset$, call a chain of P with ordered elements $a_1, a_2, \ldots, a_{2k+1}$, an *alternating chain* (with respect to P^+, P^-) if $a_i \in P^+$ for i odd, and $a_i \in P^-$ for i even. Finally, for any $S \subseteqq P$, call $|S \cap P^+| - |S \cap P^-|$ the *surplus* of S (with respect to P^+, P^-).

Corollary 4.4. *Given a partition* P^+, P^- *of the poset P, the maximum surplus of a convex set of P is equal to the minimum number of alternating chains which cover* P^+ *and are packed in* P^-.

Proof. Let $G = (P, E)$ be the comparability graph of P. Any alternating vector A_i of G can be considered to be the alternating vector of a path whose node set is the support of A_i. Define c by $c_v = 1$ for $v \in P^+$, $c_v = -1$ for $v \in P^-$. The corollary follows then from the duality theorem fulfilled with integer primal and dual solutions (use also (ii) of Theorem 4.3). ∎

Corollary 4.4 appears to be a close analogue to Dilworth's theorem [1], stating that the maximum cardinality of an antichain of a poset P is equal to the minimum number of chains which are necessary to cover P. To obtain the latter from this corollary, introduce for any $a, b \in P$ such that $a > b$, an element $\bar{a}$ and the relations $a > \bar{a} > b$ (together with all relations deduced by transitivity), and let P^- be the set of the introduced elements and $P^+ = P$.

Remark 4.5. It is easy to extend the algorithms to the case where G has circuits, and solve thereby "maximize cx s.t. $Ax \leqq 1, Rx \leqq 0, x \geqq 0$", and its dual. In fact, the primal algorithm remains unchanged. The dual algorithm involves a flow decomposition into paths and circuits of $\hat{G}$. Some care must be given to the fact that to a path (circuit) of $\hat{G}$ might now correspond a non-elementary path (circuit), whose "alternating vector" b has to be suitably decomposed into alternating vectors A_i and R_j's [7]. Of course, the algorithm again yields an integer optimal dual solution if c is integer, proving that the system in (1.2) is tdi. However, the rows corresponding to the support of the solution might not all conform to c.

We conclude by observing that the algorithms provide an alternative proof of (1.2) and total dual integrality of the corresponding system. The same can be done for (1.1) by deriving easy good algorithms for the lp's associated to (1.1) [7]. Nevertheless, proving the property of total dual integrality and the integrality of a polyhedron through non-algorithmic means, and possibly classifying the polyhedron, is of interest in itself. Section 2 shows that (1.1) is essentially a lattice polyhedron. In [9], we show that, for a graph without circuits, (1.2) is member of an another class of integer polyhedra whose systems are tdi, which can be called *switching paths polyhedra,* and which are a generalization of Hoffman's polyhedra of [12].

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